Explicit Asymptotic Magnetic Surfaces

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Toroidal magnetic fields $\boldsymbol{B} = \boldsymbol{B}_0 + \varepsilon \boldsymbol{B}_1 + \dots$, where the unperturbed field $\boldsymbol{B}_0$ has closed lines of force, are considered. Single-valued formal solutions $F = F_0 + \varepsilon F_1 + \dots$ of the equation $\boldsymbol{B} \cdot \nabla F = 0$ are explicitly determined.

If a magnetic field $\boldsymbol{B}$ only slightly deviates from a fundamental field $\boldsymbol{B}_0$ without rotational transform, then the field may be written in the form

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1,$$

where $\boldsymbol{B}_1$ is everywhere small compared with $\boldsymbol{B}_0$. The asymptotic magnetic surfaces [1] are single-valued formal solutions F of the equation

$$\boldsymbol{B} \cdot \nabla F = 0. \tag{1}$$

If one writes

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots, \tag{2}$$

then the surfaces $F_0 = \text{const}$ are not uniquely determined by the equation

$$\boldsymbol{B} \cdot \nabla F_0 = 0$$

but follow from solubility conditions in higher order. It has been shown in [2] that for analytical fields the asymptotic surfaces (2) uniquely exist in all orders. In this paper the F's are explicitly computed.

We use the notation of [3] rather than that of [2] and express the solenoidal property by

$$\boldsymbol{B}_0 = \nabla \psi \times \nabla \chi$$

and

$$\boldsymbol{B}_1 = \nabla u \times \nabla \psi - \nabla v \times \nabla \chi .$$

Here, ψ, χ are single-valued functions of the position vector $\boldsymbol{x}$, while u, v are, in general, multivalued. The functions $\psi(\boldsymbol{x})$, $\chi(\boldsymbol{x})$ are well suited to serve as coordinates. In addition, we choose a third function $\sigma(\boldsymbol{x})$ which increases monotonically along the field lines $\boldsymbol{B}_0$ with total increase 1. Furthermore, let the functional determinant

$$D = (\nabla \psi \times \nabla \chi) \cdot \nabla \sigma$$

be everywhere positive. In these coordinates Eq. (1) reads

$$F_{,\sigma} + v_{,\sigma} F_{,\psi} + u_{,\sigma} F_{,\chi} - (u_{,\chi} + v_{,\psi}) F_{,\sigma} = 0 , \tag{3}$$

where comma and subscript denote partial derivatives.

Let the increase of the functions u, v be

$$u^* = \int_{\sigma_0}^{\sigma_0+1} u_{,\sigma} \, \mathrm{d}\sigma , \quad v^* = \int_{\sigma_0}^{\sigma_0+1} v_{,\sigma} \, \mathrm{d}\sigma ,$$

respectively. Because of

$$\Big(\int_{\psi = \text{const}} \boldsymbol{B} \cdot \mathrm{d}^2 \boldsymbol{f} \Big)_{,\chi} = v^* ,$$

$$\Big(\int_{\chi = \text{const}} \boldsymbol{B} \cdot \mathrm{d}^2 \boldsymbol{f} \Big)_{,\psi} = u^* ,$$

these functions represent fluxes across the unperturbed field lines. The left-hand sides of the equations

$$D^{-1} \boldsymbol{B}_1 \cdot \nabla \psi = v_{,\sigma} ,$$
$$D^{-1} \boldsymbol{B}_1 \cdot \nabla \chi = u_{,\sigma} ,$$
$$D^{-1} \boldsymbol{B}_1 \cdot \nabla \sigma = - u_{,\chi} - v_{,\psi}$$

are single-valued, i.e. periodic in σ. Thus, these equations imply

$$\frac{\partial u^*}{\partial \sigma_0} = \frac{\partial v^*}{\partial \sigma_0} = u^*_{,\chi} + v^*_{,\psi} = 0 . \tag{4}$$

Equations (4) are satisfied if we put

$$v^* = U_{,\chi} , \quad u^* = - U_{,\psi} , \tag{5}$$

so that the increase of the functions u, v is solely described by the function $U(\psi, \chi)$.

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Assuming series of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots,$$
$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \dots \tag{6}$$

and for the increases

$$U = \varepsilon U_1 + \varepsilon^2 U_2 + \dots, \tag{7}$$

the expansions (2), (6) are substituted in Equation (3). This yields in lowest order

$$F_{0,\sigma} = 0$$

with the solution

$$F_0 = G(\psi, \chi)$$

and in higher order

$$\begin{aligned} f_{n,\sigma} = \sum_{k=1}^{n} [& (u_{k,\chi} + v_{k,\psi}) f_{n-k,\sigma} \\ & - v_{k,\sigma}(f_{n-k,\psi} + g_{n-k,\psi}) \\ & - u_{k,\sigma}(f_{n-k,\chi} + g_{n-k,\chi})], \end{aligned}$$
$$n = 1, 2, 3, \dots \tag{8}$$

with

$$F_n = f_n(\psi, \chi, \sigma) + g_n(\psi, \chi),$$
$$g_0 = G, \quad f_0 = 0.$$

For $n = 1$ we have

$$f_{1,\sigma} = -v_{1,\sigma} G_{,\psi} - u_{1,\sigma} G_{,\chi}. \tag{9}$$

Here, two cases are distinguished: a) $U_1 \not\equiv 0$, b) $U_1 \equiv 0$.

a) In this case the solubility condition for Eq. (9) is

$$0 = -U_{1,\chi} G_{,\psi} + U_{1,\psi} G_{,\chi},$$

which can be satisfied by putting

$$G(\psi, \chi) = G(U_1). \tag{10}$$

Equation (9) can be integrated in the form

$$f_1 = -v_1 G_{,\psi} - u_1 G_{,\chi}, \tag{11}$$

which, with the choice (10), is single-valued.

In order to determine g_1 we go on to consider order ε^2 of Eq. (3), i.e. $n = 2$ in Eq. (8)

$$\begin{aligned} f_{2,\sigma} = & (u_{1,\chi} + v_{1,\psi}) f_{1,\sigma} - v_{1,\sigma}(f_{1,\psi} + g_{1,\psi}) \\ & - u_{1,\sigma}(f_{1,\chi} + g_{1,\chi}) - v_{2,\sigma} G_{,\psi} \\ & - u_{2,\sigma} G_{,\chi}, \end{aligned} \tag{12}$$

implying that

$$\begin{aligned} 0 = & \int_{\sigma_0}^{\sigma_0+1} \mathrm{d}\sigma [v_{1,\sigma}(v_1 G_{,\psi\psi} - u_{1,\chi} G_{,\psi} + u_{1,\psi} G_{,\chi} + u_1 G_{,\psi\chi}) \\ & + u_{1,\sigma}(v_{1,\chi} G_{,\psi} - v_{1,\psi} G_{,\chi} + v_1 G_{,\psi\chi} + u_1 G_{,\chi\chi})] - v_1^* g_{1,\psi} - u_1^* g_{1,\chi} - v_2^* G_{,\psi} - u_2^* G_{,\chi} \\ = & \frac{\mathrm{d}G}{\mathrm{d}U_1} (U_{1,\psi} \partial_\chi - U_{1,\chi} \partial_\psi) [\int \bar{u}_{1,\sigma} \bar{v}_1 \mathrm{d}\sigma - U_{1,\chi} \int \bar{u}_1 \mathrm{d}\sigma - U_{1,\psi} \int \bar{v}_1 \mathrm{d}\sigma] \\ & + (U_{1,\psi} \partial_\chi - U_{1,\chi} \partial_\psi) g_1 - \frac{\mathrm{d}G}{\mathrm{d}U_1} (U_{1,\psi} \partial_\chi - U_{1,\chi} \partial_\psi) U_2, \end{aligned} \tag{13}$$

where

$$\bar{u}_1 = u_1 - u_1^* \sigma, \quad \bar{v}_1 = v_1 - v_1^* \sigma$$

are single-valued and Eqs. (5), (7), (10) have been used. The form Eq. (13) of the solubility condition, which is tedious but straightforward to derive, has the advantage that it can readily be integrated.

$$\begin{aligned} g_1 = \frac{\mathrm{d}G}{\mathrm{d}U_1} [& U_2 - \int \bar{u}_{1,\sigma} \bar{v}_1 \mathrm{d}\sigma \\ & + U_{1,\chi} \int \bar{u}_1 \mathrm{d}\sigma + U_{1,\psi} \int \bar{v}_1 \mathrm{d}\sigma]. \end{aligned} \tag{14}$$

The formulae (10), (11), (14) are the main analytical results for case (a).

(b) Since, for $U_1 \equiv 0$, u_1, v_1 are single-valued, there is no solubility condition in first order and thus G in Eq. (11) is a free function which is determined by the solubility condition for Eq. (12):

$$\begin{aligned} 0 = & \int_{\sigma_0}^{\sigma_0+1} \mathrm{d}\sigma [(v_1 f_{1,\sigma})_{,\psi} + (u_1 f_{1,\sigma})_{,\chi}] \\ & - v_2^* G_{,\psi} - u_2^* G_{,\chi} \\ = & (G_{,\psi} \partial_\chi - G_{,\chi} \partial_\psi) H, \end{aligned} \tag{15}$$

where

$$H(\psi, \chi) = -U_2 + \int u_{1,\sigma} v_1 \mathrm{d}\sigma. \tag{16}$$

Equation (15) is satisfied by

$$G(\psi, \chi) = G(H). \tag{17}$$

For f_2 we obtain

$$\begin{aligned} f_2 = & -u_{1,\chi} v_1 G_{,\psi} + v_1 u_{1,\psi} G_{,\chi} + \tfrac{1}{2} u_1^2 G_{,\chi\chi} \\ & + \tfrac{1}{2} v_1^2 G_{,\psi\psi} + u_1 v_1 G_{,\psi\chi} \end{aligned}$$

$$+ (h,_\chi - v_2) G,_\psi - (h,_\psi + u_2) G,_\chi - v_1 g_{1,\psi} - u_1 g_{1,\chi},$$

where

$$h = \int^{\sigma}_{\sigma_0} u_{1,\sigma'} v_1 \,\mathrm{d}\sigma' .$$

For determination of g_1 order ε^3 of Eq. (3) has to be considered, i.e. $n=3$ in Eq. (8):

$$\begin{aligned} f_{3,\sigma} = {} & (u_{1,\chi} + v_{1,\psi}) f_{2,\sigma} + (u_{2,\chi} + v_{2,\psi}) f_{1,\sigma} \\ & - v_{1,\sigma}(f_{2,\psi} + g_{2,\psi}) \\ & - v_{2,\sigma}(f_{1,\psi} + g_{1,\psi}) - v_{3,\sigma} G,_\psi \\ & - u_{1,\sigma}(f_{2,\chi} + g_{2,\chi}) \\ & - u_{2,\sigma}(f_{1,\chi} + g_{1,\chi}) - u_{3,\sigma} G,_\chi . \end{aligned} \tag{18}$$

Equation (18) yields the solubility condition

$$\begin{aligned} 0 = {} & \int_{\sigma_0}^{\sigma_0+1} \mathrm{d}\sigma [(v_1 f_{2,\sigma}),_\psi + (u_1 f_{2,\sigma}),_\chi + (u_{2,\chi} + v_{2,\psi}) f_{1,\sigma} - v_{2,\sigma} f_{1,\psi} - u_{2,\sigma} f_{1,\chi}] \\ & - v_2^* g_{1,\psi} - u_2^* g_{1,\chi} - v_3^* G,_\psi - u_3^* G,_\chi \\ = {} & - (H,_\psi \partial_\chi - H,_\chi \partial_\psi) g_1 + \frac{\mathrm{d}G}{\mathrm{d}H} (H,_\psi \partial_\chi - H,_\chi \partial_\psi)(b,_\psi - a,_\chi) \\ & + \frac{\mathrm{d}G}{\mathrm{d}H} (H,_\psi \partial_\chi - H,_\chi \partial_\psi) \int \mathrm{d}\sigma (v_1 u_{2,\sigma} - u_1 v_{2,\sigma}) - \frac{\mathrm{d}G}{\mathrm{d}H} (H,_\psi \partial_\chi - H,_\chi \partial_\psi) U_3 , \end{aligned} \tag{19}$$

where

$$a = \tfrac{1}{2} \int \mathrm{d}\sigma v_{1,\sigma} u_1^2 , \quad b = \tfrac{1}{2} \int \mathrm{d}\sigma u_{1,\sigma} v_1^2 . \tag{20}$$

Equation (19) is integrated with

$$g_1 = \frac{\mathrm{d}G}{\mathrm{d}H} [b,_\psi - a,_\chi - U_3 + \int \mathrm{d}\sigma (v_1 u_{2,\sigma} - u_1 v_{2,\sigma})] , \tag{21}$$

which completes the first-order result for case (b). The solution for case (b) is thus represented by the formulae (16), (17), (11), (20), and (21).

Finally, we give an example. As zeroth-order field we choose the toroidal vacuum field, given by $1/r$, where r, φ, z are cylindrical coordinates, and represent this field by

$$\psi = -\ln r , \quad \chi = z .$$

The third coordinate is given by

$$\sigma = \varphi / 2\pi .$$

If a perturbing vacuum field $\boldsymbol{B}_1$ with n periods around the torus ($n \neq 0$) is given by its scalar potential Φ_n, satisfying Laplace's equation,

$$\Phi_n = C_n \cos n\varphi + S_n \sin n\varphi ,$$

we find for the representation in terms of u, v

$$u_{1n} = u_c \cos n\varphi + u_s \sin n\varphi ,$$

$$v_{1n} = v_c \cos n\varphi + v_s \sin n\varphi ,$$

with

$$u_c = -\frac{r^2}{n} S_{n,z} ,$$

$$u_s = \frac{r^2}{n} C_{n,z} ,$$

$$v_c = \frac{r}{n} S_{n,r} ,$$

$$v_s = -\frac{r}{n} C_{n,r} ,$$

where use has been made of Laplace's equation for C_n, S_n. In the case $n=0$, the perturbing potential C is related to u and v by

$$u_1 = r^2 C,_z \varphi , \quad v_1 = -r C,_r \varphi ,$$

so that

$$u_1^* = 2\pi r^2 C,_z , \quad v_1^* = -2\pi r C,_r ,$$

and

$$U = -2\pi r \int_0^z C,_r \mathrm{d}z' + \int_{r_0}^{r} 2\pi r' C,_z(r', 0)\,\mathrm{d}r' . \tag{22}$$

Thus, u and v can be explicitly determined if an arbitrary field is given by its Fourier-analyzed potential; the expansions eqs. (6), (7) are then trivial and the results Eqs. (10), (11), (14), (16), (17), (20) are explicit.

The most interesting case for stellarator applications is a first-order perturbing field with vanishing mean value with respect to φ and a second-order perturbation independent of φ. If we consider a single n, the explicit results are

$$H(r, z) = \frac{r^3}{n} \pi (C_{n,z} S_{n,r} - S_{n,z} C_{n,r}) - U_2 ,$$

where U_2 is given by Eq. (22), and with $G \equiv H$ one has

$$f_1 = v_{1n} r H_{,r} - u_{1n} H_{,z} , \quad g_1 \equiv 0 .$$

To summarize it is concluded that, with u and v for the representation of the perturbing field, the results become explicit with respect to the variables ψ and χ. Further specialization to the simplest toroidal field renders the results explicit with respect to the toroidal coordinate, too, if Fourier analysis is applied to this coordinate. The method can for example, be used to compare numerical and analytical (as obtained here) vacuum field magnetic surfaces with small rotational transform and be applied to small-β expansions which do not use any expansions with respect to geometrical parameters such as the inverse aspect ratio and the form of the flux surfaces.

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